

Short-time behavior of the kinetic spherical model with long-ranged interactions^{*}

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Abstract. The kinetic spherical model with long-ranged interactions and an arbitrary initial order m_0 quenched from a very high temperature to T ($\leq T_c$) is solved. In the short-time regime, the bulk order increases with a power law in both the critical and phase-ordering dynamics. To the latter dynamics, a power law for the relative order $m_r \sim -t^{-k}$ is found in the intermediate time-regime. The short-time scaling relations of small m_0 are generalized to an arbitrary m_0 and all the time larger than t_{mic} . The characteristic functions $\varphi(b, m_0)$ for the scaling of m_0 and $\epsilon(b, T')$ for $T' = T/T_c$ are obtained. The crossover between scaling regimes is discussed in detail.

PACS. 64.60.Ht Dynamic critical phenomena – 64.60.Cn Order-disorder transformations; statistical mechanics of model systems – 05.70.Ln Nonequilibrium and irreversible thermodynamics

1 Introduction

In recent years, the universal scaling in non-equilibrium states have attracted much attention. The phase-ordering process (POP) [1] and the short-time critical dynamics (SCD) [2] are two fruitful examples.

In the critical dynamics, the short-time phenomena are those which happen at the times just after a microscopic time-scale t_{mic} needed for a system to forget its microscopic details, and much smaller than the macroscopic time scale $t_{\text{mac}} \sim \tau^{-\nu z}$. Since the pioneer work of Janssen *et al.* [3], universal short-time scaling has been found in a variety of different models (for a recent review, see [4]). The short-time singularity of the response propagator is governed by a new critical exponent θ . The order increases in the short-time regime with a power law $t^{\theta'}$ where θ' is another characteristic exponent. It is believed that the singularity of the temporal correlation is essential to the short-time scaling. Therefore the scaling can emerge in the short-time regime of the evolution eventhough all spatial correlations are still short-ranged.

The dynamics to be considered here has no conservation law, which is sometimes called model A [5]. All critical exponents as well as the critical point can be fixed in the short-time regime [6–8]. This is important especially for Monte-Carlo simulations because at the critical temperature a system needs infinite time to relax to equilibrium

due to the critical slowing down. It is good to avoid doing measurements in equilibrium states. Therefore, it is interesting to investigate the connection between the short-time regime and the long-time regime and the crossover of scaling patterns from one regime to another.

The power law increase of order in the SCD is valid only in the case of a small initial order-parameter m_0 . The general case of an arbitrary m_0 has been discussed in [9]. The initial state with non-zero m_0 is off-critical. A characteristic function must be introduced to describe the moving of m_0 in the scale transformation. With the characteristic function, the scaling law can be extended to a large m_0 and all $t \gg t_{\text{mic}}$. As far as we know, only the numerical evidence has been found for this characteristic function. In this paper, the soluble kinetic spherical model (KSM) will provide a concrete example.

Instead of a quench to the critical temperature T_c in the SCD, the system in the POP is quenched to a temperature lower than the critical one. For a system quenched from a symmetric initial state, the final equilibrium is never achieved since the ergodicity is broken in the thermodynamics. Instead, the length-scale of ordered regions grows with time as the different broken-symmetry phases compete to select the equilibrium state. This temporal singularity should also be the essence of the ordering scaling. It has been found both experimentally and theoretically [10] that the POP at late times can be described by a single characteristic length-scale $L(t) \sim t^\rho$, reflecting the self-similarity of domain patterns at different times.

If the initial state breaks the symmetry, as the case of $m_0 \neq 0$ to be discussed here, the equilibrium can be achieved. A natural question is how long the phase-ordering scaling will last. The non-zero m_0 generates

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a time-scale t_i which characterizes the most interesting short-time regime where we will see new scaling patterns. Since $t_i \rightarrow \infty$ as $m_0 \rightarrow 0$, the so-called short-time regime in fact can persist to a very long time.

The SCD and the POP are governed by two different fixed points, *i.e.*, T_c and $T = 0$ respectively, therefore they have different scaling laws. It will be interesting to see how the POP crosses over to the SCD as T grows from 0 to T_c . In the context of POP, besides t_i there is another time-scale t_μ given by the non-zero temperature. In this paper, we will introduce a characteristic function to describe the temperature dependence of the process. Since the critical exponents can not change smoothly, there must be a crossover region that separates the SCD scaling domain from the scaling domain of POP. We will estimate the domain boundary through a self-consistency of the solution.

To obtain explicit analytical results, we confine ourselves to the soluble KSM with or without long-ranged interactions. The KSM is a generalization to the static spherical model [11], to which a model A dynamics is given by the Langevin equation. The static spherical model has been proven to be equivalent to the large- n limit n -vector model (LLNM) in static properties [12]. When the dynamics is included, they should be still equivalent in the thermodynamic limit since the fluctuation of $\langle \phi^2 \rangle$ in the LLNM can be neglected. The LLNM being a soluble kinetic model with a dimensionality higher than one has been extensively studied by many authors. The SCD of the LLNM with the short-ranged interaction and small m_0 is well-known [3, 13, 14]. To the symmetric POP, *i.e.*, $m_0 = 0$, many results have been obtained for the model with either the short-ranged or the long-ranged interaction [15–18]. The non-symmetric case firstly was addressed by Bray and Kissner [19] for the short-ranged interaction. We will use the language of KSM to present our new results originated from the non-symmetry initial state, *i.e.*, an arbitrary m_0 . The KSM has the domain-wall picture which is helpful for understanding the ordering process and the crossover from the POP to the SCD.

In this paper, we will concentrate on the consequences of non-symmetric initial states. The time-dependent order parameter $m(t)$ as well as the response propagator and correlation function are calculated. The dynamic behavior of the symmetric KSM with short-ranged interaction in the long-time regime was already studied in [20], where no trace of the initial condition is left. In the case of non-zero m_0 , the order-parameter has non-equilibrium values in the short-time regime after the system is quenched from a very high temperature. It will show rich scaling patterns governed by the double time-scales t_i and t_μ .

It has been known that the critical exponents can be modified by the interaction range. How the characteristic critical exponent θ or θ' depends on the interaction range has not been explicitly shown. But from the general consideration of that the longer interaction range and higher dimension are both in favor of the spacial correlation, the long interaction range will decrease the exponent θ' . We will show that is just the case in the KSM. Particularly we show that the interaction-range parameter σ and the

dimensionality d enter the time-dependent order-parameter $m(t)$ always in the combination d/σ .

The remainder of this paper is organized as follows. In Section 2 the kinetic spherical model is introduced. Laplace transformation to $m(t)$ is established in Section 3. Solutions and scaling behaviors at $T = T_c$ and $T < T_c$ are obtained in Sections 4 and 5, respectively. In Section 6, a concrete example of $d/\sigma = 3/2$ is given. Our conclusions and discussions will be given in the last section.

2 The model

The Hamiltonian of the spherical model is

$$H = \frac{\alpha}{2} \sum_i S_i^2 - \frac{\beta}{2} \sum_{ij} J_{ij} S_i S_j \quad (1)$$

with the constraint

$$\sum_i S_i^2 = N \quad (2)$$

where i, j are labels of lattice sites, N is the total number of spins; $\beta = \frac{1}{k_B T}$. In the dynamic process, α is a time-dependent Lagrange multiplier corresponding to the constraint. Joyce [21] first studied the static spherical model with long-ranged ferromagnetic interactions. In a d -dimensional lattice,

$$J_{ij} = J_0 r_{ij}^{-(d+s)} / \sum_j r_{ij}^{-(d+s)}$$

with $0 < s < 2$ for long-ranged interactions, while $s > 2$ for short-ranged interactions. Where r_{ij} is the distance between the sites i and j .

The Langevin equation for this constrained spin system is

$$\frac{\partial S_i}{\partial t} = -\lambda \alpha S_i + \lambda \beta \sum_j J_{ij} S_j + \eta_i \quad (3)$$

where λ is the kinetic coefficient and η_i being a Gaussian white noise characterized by

$$\langle \eta_i(t) \rangle = 0, \quad \langle \eta_i(t) \eta_j(t') \rangle = 2\lambda \delta_{ij} \delta(t - t') \quad (4)$$

where angle brackets mean average over the noise.

Since the constraint is applied to the whole dynamic process, at any time t , there is a consistency condition

$$\left\langle \sum_i S_i \frac{\partial S_i}{\partial t} \right\rangle = 0. \quad (5)$$

The time-dependent order-parameter $m(t) = \frac{1}{N} \langle \sum_i S_i \rangle$ is in the translationally invariant case equal to $\langle S_i \rangle$. Taking the average over the noise in (3), one has

$$\frac{\partial m(t)}{\partial t} = -\lambda \tau(t) m(t) \quad (6)$$

where $\tau(t) = \alpha(t) - \beta J_0$.

The fluctuations of spins are defined to be $\tilde{S}_i = S_i - m(t)$. In the momentum space, one has

$$\tilde{S}(p, t) = \frac{1}{\sqrt{N}} \sum_i \tilde{S}_i e^{i\mathbf{p} \cdot \mathbf{r}_i}, \quad \eta(p, t) = \frac{1}{\sqrt{N}} \sum_i \eta_i e^{i\mathbf{p} \cdot \mathbf{r}_i},$$

with \mathbf{r}_i the position vector of site i , and

$$J(p) = \sum_j J_{ij} e^{i\mathbf{p} \cdot \mathbf{r}_{ij}}$$

with $J(0) = J_0$. Corresponding to (3), the dynamic equation for $\tilde{S}(p, t)$ is

$$\frac{\partial \tilde{S}(p, t)}{\partial t} = -\lambda(\tau(t) + \Delta(p))\tilde{S}(p, t) + \eta(p, t) \quad (7)$$

where $\Delta(p) = \beta(J_0 - J(p))$. The consistency condition gives

$$\begin{aligned} \tau(t) &= \tau_{\text{sub}} + \beta J_0 [m^2(t) - 1] \\ &+ \frac{\beta}{N} \sum_p J(p) \langle \tilde{S}(-p, t) \tilde{S}(p, t) \rangle \\ &+ \frac{1}{\lambda N} \sum_p \langle \tilde{S}(-p, t) \eta(p, t) \rangle \end{aligned} \quad (8)$$

where the first term comes from the mass subtraction which guarantees $\tau_c(\infty) = 0$ at the critical point. Recalling that the equilibrium correlation has a zero pole at the critical temperature, one has

$$\langle \tilde{S}(-p, \infty) \tilde{S}(p, \infty) \rangle|_{\beta_c} = 1/\Delta(p)|_{\beta_c}.$$

It defines the critical temperature as given latter in (13). Substituting it into (8) at the critical temperature, one can find $\tau_{\text{sub}} = 1$ for the Ito prescription with which the last term of (8) is zero due to causality. If the Stratonovich prescription is used, the last term of (8) is 1 and $\tau_{\text{sub}} = 0$. Through the paper, Ito prescription will be used.

By solving (7), it is not difficult to obtain the response propagator

$$\begin{aligned} G_p(t, t') &= \frac{1}{2\lambda} \langle \tilde{S}(-p, t) \eta(p, t') \rangle \\ &= \Theta(t - t') e^{-\lambda \Delta(p)(t-t') - \lambda \int_{t'}^t dt'' \tau(t'')} \end{aligned} \quad (9)$$

with the Heaviside step function $\Theta(t - t') = 1$ for $t > t'$, otherwise $\Theta(t - t') = 0$; and the full correlation function (correlation function including the initial correlation)

$$\begin{aligned} \tilde{C}_p(t, t') &= \langle \tilde{S}(p, t) \tilde{S}(-p, t') \rangle \\ &= \langle \tilde{S}(p, 0) \tilde{S}(-p, 0) \rangle G_p(t, 0) G_{-p}(t', 0) + C_p(t, t') \end{aligned} \quad (10)$$

with the correlation function

$$C_p(t, t') = 2\lambda \int_0^\infty dt'' G_p(t, t'') G_p(t', t'') \quad (11)$$

where $\langle \tilde{S}(p, 0) \tilde{S}(-p, 0) \rangle$ is given by the initial state. Consider a system initially at a temperature $T \gg T_c$ with a given initial order-parameter m_0 . Since initial correlations are short-ranged, one has $\langle \tilde{S}(p, 0) \tilde{S}(-p, 0) \rangle = (1 - m_0^2)$.

3 Laplace transformation

Introducing $f(t) = m^{-2}(t)$, from (6) and (8) one obtains a linear integrodifferential equation for $f(t)$

$$\begin{aligned} \frac{\partial f(t)}{\partial t} &= 2\lambda\beta J_0 - 2\lambda(\beta J_0 - 1)f(t) \\ &+ \frac{2\lambda\beta}{N} \sum_p J(p) \tilde{C}_p(t, t) f(t). \end{aligned} \quad (12)$$

For a d -dimensional cubic lattice,

$$J(p) = J_0 C_{d,s}(p) / C_{d,s}(0)$$

with $C_{d,s}(p) = \sum_l |\mathbf{l}|^{-(d+s)} \cos(\mathbf{l} \cdot \mathbf{p})$ and $\mathbf{l} \cdot \mathbf{p} = l_1 p_1 + \dots + l_d p_d$. The critical temperature is given by [21]

$$\beta_c J_0 = \frac{1}{N} \sum_p [1 - C_{d,s}(p) / C_{d,s}(0)]^{-1}. \quad (13)$$

Define

$$w(x) = \frac{1}{N} \sum_p \frac{1}{x + \Delta(p) / (\beta J_0)} \quad (14)$$

then $\beta_c J_0 = w(0)$. By Laplace transformation

$$F(q) = \int_0^\infty dt f(t) e^{-qt}.$$

Equation (12) is transformed to

$$F(q) = \frac{\frac{\beta J_0}{q} + \frac{1}{2\lambda}(m_0^{-2} - 1)w(\frac{q}{2\lambda\beta J_0})}{\beta J_0 - w(\frac{q}{2\lambda\beta J_0})}. \quad (15)$$

Using properties of Laplace-transformation, it is easy to obtain

$$\lim_{q \rightarrow \infty} qF(q) = m_0^{-2} \quad (16)$$

$$\lim_{q \rightarrow 0} qF(q) = m^{-2}(\infty) = (1 - T/T_c)^{-1}. \quad (17)$$

Hence the initial condition is satisfied, and the infinite time limit correctly recovers the equilibrium result. From (17), one can also extract the critical exponent $\beta = 1/2$.

For small p , one has

$$1 - C_{d,s}(p) / C_{d,s}(0) = Cp^\sigma \quad (18)$$

where the constant C depends only on d and s . The parameter σ denotes an effective potential-range defined by [22], with $\sigma = s$ for $0 < s < 2$ and $\sigma = 2$ for $s > 2$.

From here after, we suppose $d/2 < \sigma < d$. For the infinite system, the sum in (14) is replaced by an integral. Combining (13) with (18), in the continuum-limit one has

$$w(x) = w(0) - Dx^{d/\sigma-1} \quad (19)$$

where the constant D again depends only on σ and d and is positive. Then (15) becomes

$$F(q) = \frac{\frac{\beta J_0}{q} + \frac{\beta J_0}{2\lambda}(m_0^{-2} - 1)}{(\beta - \beta_c)J_0 + D\left(\frac{q}{2\lambda\beta J_0}\right)^{d/\sigma-1}} - \frac{1}{2\lambda}(m_0^{-2} - 1). \quad (20)$$

Since the microscopic details have been skipped in the continuum limit, (20) does not give the correct initial value. Instead, it gives $f(0) = \infty$ corresponding to $m(0) = 0$. Therefore (20) is only valid for t larger than a microscopic time-scale t_{mic} . The last term of (20) only has contribution at $t = 0$, therefore it can be dropped for the regime $t > t_{\text{mic}}$.

Notably, if one rescales the kinetic coefficient λ and the initial order m_0 as following

$$\lambda' = \lambda D_c^{\frac{1}{1-d/\sigma}} \quad (21)$$

$$m'_0 = \left[D_c^{\frac{1}{1-d/\sigma}} (m_0^{-2} - 1) + 1 \right]^{-1/2} \quad (22)$$

with $D_c = D/(\beta_c J_0)^{d/\sigma}$, then $F(q)$ depends on the dimension d and the interaction-range parameter σ through their ratio $k = d/\sigma$, *i.e.*, for $t > t_{\text{mic}}$,

$$F(q) = \frac{\frac{1}{q} + \frac{1}{2\lambda'}(m'^{-2}_0 - 1)}{\mu + (1 - \mu)^k \left(\frac{q}{2\lambda'}\right)^{k-1}} \quad (23)$$

where $\mu = (T_c - T)/T_c$ is the reduced temperature. D_c is a non-universal constant depending on d , σ and the regularization scheme. As one will see in the next section, (22) follows from a rescaling of time by a factor $D_c^{\frac{1}{1-k}}$.

Equation (23) is the starting point of the following discussions.

4 T = T_c

At the critical point, $\mu = 0$, the Laplace back-transformation of (23) is easily carried out. The solution for the order parameter at the critical point is

$$m(t) = \pm \left[\frac{\Gamma(k)}{(2\lambda')^{k-1} t_i} \right]^{1/2} (1 + t/t_i)^{-1/2} t^{1-\frac{k}{2}} \quad (24)$$

where t_i is a characteristic time-scale of the short-time regime, which depends on the initial condition,

$$t_i = \frac{(k-1)(m_0^{-2} - 1)}{2\lambda} = \frac{(k-1)(m'^{-2}_0 - 1)}{2\lambda'}. \quad (25)$$

It is invariant in the combined transformation of (21, 22). The sign of $m(t)$ can be found from the symmetry of the initial state. The $|m(t)|$ firstly increases then it relaxes to zero. The maximum is at $t = \frac{2-k}{k-1} t_i$, it is proportional to t_i .

For small enough m_0 , the t_i could be much larger than t_{mic} . Then in the regime $t_{\text{mic}} < t \ll t_i$ one can observe the power-law order increasing suggested by Janssen *et al.* [3],

$$m(t) \sim t^{\theta'} \quad (26)$$

with the exponent $\theta' = 1 - \frac{k}{2}$. This regime is called the critical initial slip. A standard explanation to the increase of the order is that the order-parameter follows a mean-field ordering process because $T_c < T_c^{\text{m.f.}}$ as long as correlations are short-ranged in the critical initial slip. However the exponent θ' is non-trivial. As we will see in the next section, it is different from the corresponding exponent for the POP.

In the long-time regime, $t \gg t_i$,

$$m(t) = \left[\frac{\Gamma(k)}{(2\lambda')^{k-1}} \right]^{1/2} t^{-\frac{k-1}{2}}. \quad (27)$$

The dependence on the initial condition disappears as expected. Comparing with the well-known nonlinear relaxation scaling law, one obtains $\frac{\beta}{\nu z} = \frac{k-1}{2}$.

When $m_0 \neq 0$, the initial state is not at the fixed point. However, as proposed by Zheng [9], the short-time critical scaling hypothesis could be generalized to an arbitrary m_0 by introducing a characteristic function of m_0 and the scaling factor b .

As the length is rescaled by a factor b , supposing t_i is rescaled by b^{-z} just like a time, then $m(t)$ given by (24) satisfies the following scaling relation

$$m(t, m_0) = b^{-\frac{\beta}{\nu}} m(b^{-z} t, \varphi(b, m_0)) \quad (28)$$

where $\varphi(b, m_0)$ is the characteristic function, which describes how the initial order changes in the scale transformation. Its explicit form can be deduced from the scaling of t_i

$$\varphi(b, m_0) = [b^{-z}(m_0^{-2} - 1) + 1]^{-\frac{1}{2}}. \quad (29)$$

The effective dimension of the initial order m_0 is defined by

$$x_0(b, m_0) = \frac{\ln[\varphi(b, m_0)/m_0]}{\ln b}. \quad (30)$$

As expected, as $m_0 \rightarrow 0$, $\varphi(b, m_0) = b^{z/2} m_0$, so that the initial order has an anomalous dimension $x_0 = \frac{z}{2}$ and the power law (26) is recovered. When m_0 takes its maximum $m_0 = 1$, $\varphi = 1$ and $x_0 = 0$. In this case, $t_i = 0$, hence only the nonlinear relaxation scaling law, $m(t) \sim t^{-\frac{\beta}{\nu z}}$, is observed. Figure 1 plots x_0 versus m_0 with $z = 2$ for $b = 2$ and $b = 4$. It is qualitatively consistent with the numerical results of Ising model [4], for which $x_0(b, m_0)$ is a monotonously decreasing function of m_0 . This appears to be reasonable by recalling that the spherical model is just a very rough approximation of the Ising model.

Substituting (24) into (6), one gets $\tau(t) = \frac{k-2}{2\lambda t} + \frac{1}{2\lambda(t+t_i)}$. This result together with (9) gives the response propagator

$$G_p(t, t') = \Theta(t - t') \left(\frac{t}{t'} \right)^{(2-k)/2} \left(\frac{t' + t_i}{t + t_i} \right)^{1/2} e^{-\bar{\lambda} p^\sigma (t-t')} \quad (31)$$

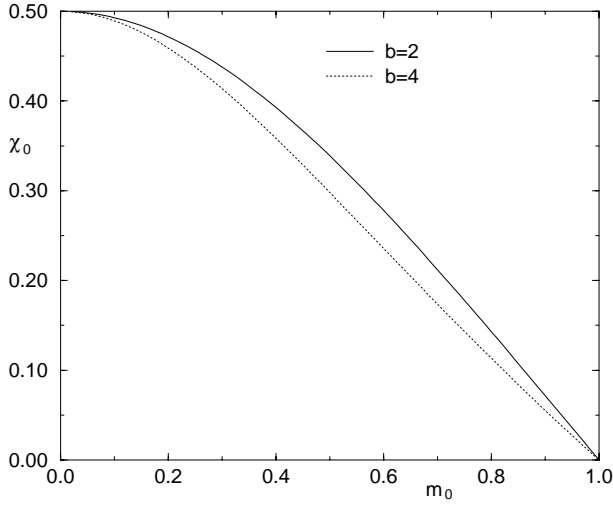


Fig. 1. The effective dimension $x_0(b, m_0)$ versus m_0 with $z = 2$. The upper line is for $b = 2$ while the lower line for $b = 4$.

where $\tilde{\lambda} = \lambda C \beta J_0$ with $\beta = \beta_c$ in this case. The correlation function (11) is given by

$$C_p(t, t') = 2\lambda t' \left(\frac{t}{t'}\right)^{\frac{2-k}{2}} \left(\frac{t' + t_i}{t + t_i}\right)^{\frac{1}{2}} e^{-\tilde{\lambda} p^\sigma (t-t')} \times \int_0^1 dy (1-y)^{k-2} \frac{t' - t'y + t_i}{t' + t_i} e^{-2\tilde{\lambda} p^\sigma t'y}. \quad (32)$$

When $t \ll t_i$ and $t' \ll t_i$, (31) has a scaling form,

$$G_p(t, t') = p^{-2+\eta+z} \left(\frac{t}{t'}\right)^\theta h(p^z(t-t'))$$

with $h(x)$ being a scaling function, as suggested in [2]. The exponents $\theta = 1 - \frac{k}{2}$, $z = \sigma$ and $\eta = 2 - \sigma$ can be read out. The values of z and η agree with the corresponding results in [23–25] and [21, 26, 27]. Two exponents θ and θ' are equal to each other. For short-ranged interactions $\sigma = 2$, they coincide with the results of LLNM [3, 28, 13, 14].

When $t = t' \ll t_i$, we get the equal-time correlation function

$$C_p(t, t) = \frac{2\lambda t e^{-2\tilde{\lambda} p^\sigma t}}{k-1} (1 + O(2\tilde{\lambda} p^\sigma t)). \quad (33)$$

It has the scaling form [3, 13–15, 29] $C_p(t, t) = p^{-2+\eta} g(p\xi_c(t))$ with $g(x)$ being another scaling function. Where $\xi_c(t) \sim t^{1/\sigma}$ is a characteristic length-scale. Physically, it should be related to the correlation length.

When $t, t' \gg t_i$ and $2\tilde{\lambda} p^\sigma t'$ large, the propagator and equal-time correlation function reads as

$$G_p(t, t') = \Theta(t-t') \left(\frac{t}{t'}\right)^{-\frac{k-1}{2}} e^{-\tilde{\lambda} p^\sigma (t-t')} \quad (34)$$

$$C_p(t, t) = \frac{p^{-\sigma}}{C\beta_c J_0} \left[1 - \frac{k-1}{2\tilde{\lambda} p^\sigma t} + O\left(\frac{1}{(\tilde{\lambda} p^\sigma t)^2}\right) \right]. \quad (35)$$

By the use of the characteristic function (29), the response propagator and correlation function have general scaling forms which are valid for the times up to t_{mic} and for an arbitrary m_0 ,

$$G_p(t, t', m_0) = p^{-2+\eta+z} \tilde{h}(p\xi_c(t), p\xi_c(t'), \varphi(p^{-1}, m_0)) \quad (36)$$

$$C_p(t, t', m_0) = p^{-2+\eta} \tilde{g}(p\xi_c(t), p\xi_c(t'), \varphi(p^{-1}, m_0)) \quad (37)$$

where \tilde{h} and \tilde{g} are scaling functions. The significance of the characteristic function is to connect the short-time regime and the long-time regime so that for each thermodynamic quantity only one scaling function is needed for the whole physical interesting time regime.

5 $T < T_c$

In the case of $T < T_c$, the system undergoes an ordering process which is governed by the fixed point $T = 0$. For large β , *i.e.*, $\mu \rightarrow 1$, one can expand the denominator of (23) to get

$$F(q) = \mu^{-1} \left[\frac{1}{q} - \frac{2\lambda'}{\mu} \left(\frac{1-\mu}{2\lambda'}\right)^k q^{k-2} - \frac{m_0'^{-2} - 1}{\mu} \left(\frac{1-\mu}{2\lambda'}\right)^k q^{k-1} \right]. \quad (38)$$

The asymptotic solution for the bulk order is obtained by the Laplace back-transformation

$$m(t) = m(\infty) \left[1 - \left(1 - \frac{t_i}{t}\right) \left(\frac{t}{t_i}\right)^{1-k} \right]^{-1/2} \quad (39)$$

where $t_\mu = \left[\frac{2\lambda'}{|\mu| \Gamma(2-k)} \left(\frac{1-\mu}{2\lambda'}\right)^k \right]^{\frac{1}{k-1}}$. Again for a fixed k , the dependence on d and σ can be absorbed by a proper rescaling of the time.

The t_μ is a temperature-dependent time-scale. The corresponding length-scale should be the domain-width $\xi_{\text{dw}} \sim t_\mu^{1/z}$. The POP scaling only could survive when ξ_{dw} is smaller than the domain-size which grows with time as $\xi \sim t^{1/z}$. That implies t_μ should be smaller than a characteristic time of the phase-ordering regime. From the consistency of (39) we do find a constraint for t_μ

$$\frac{t_\mu}{t_i} < \frac{k^{\frac{k}{k-1}}}{k-1}.$$

Replacing t_μ by its explicit expression, one has

$$\left(1 - \frac{T}{T_c}\right) \left(\frac{T_c}{T}\right)^k > \frac{1}{k\Gamma(2-k)} \left(\frac{k-1}{2\lambda'kt_i}\right)^{k-1}. \quad (40)$$

The r.h.s is a positive constant. The l.h.s is a monotonously decreasing function with respect to T ,

which has zero value at $T = T_c$. Hence there is a boundary temperature $T_b < T_c$. The ordering scaling appears in the regime $0 < T < T_b$. The smaller m_0 , the larger T_b . It tends to T_c as $m_0 \rightarrow 0$.

As $t \ll t_i$, one has

$$m(t) = m(\infty) \left[1 + \left(\frac{t'_i}{t} \right)^k \right]^{-\frac{1}{2}} \quad (41)$$

where $t'_i = t_\mu(t_i/t_\mu)^{1/k}$ is a time-scale that characterizes the short-time ordering behavior. In the symmetry case, $t'_i = \infty$ so $m(t) = 0$. No information about the POP can be extracted from the bulk order-parameter. With a non-symmetric initial state however, the process has various scaling behaviors manifested by $m(t)$ due to the competition of two time-scales. From the self-consistency condition mentioned above, one can show that in the scaling regime, t_i is at least larger than $t'_i/2$. There will be three different scaling regimes.

(a) $t_{\text{mic}} < t \ll t'_i$ and t_i : this is the extensively studied regime in the symmetric POP. In literatures, sometimes it is called the late time stage. There is no contradiction because in the symmetry case t_i and t'_i are infinite. The late-time simply means $t > t_{\text{mic}}$.

For small m_0 , t'_i and t_i can be much larger than t_{mic} . Then in this regime, one can observe a power law increase of the order-parameter

$$m(t) = m(\infty) \left(\frac{t}{t'_i} \right)^{\theta'_T} \quad (42)$$

with $\theta'_T = k/2$. This is the initial increase in the POP.

The response propagator when $t \ll t'_i$ and t_i is

$$G_p(t, t') = \Theta(t - t') \left(\frac{t}{t'} \right)^{\theta_T} e^{-\tilde{\lambda} p^\sigma (t-t')} \quad (43)$$

with $\theta_T = k/2$. We note that expressions similar to (42, 43) have been appeared in [19,29] at $\sigma = 2$. In [19,29] $2\theta_T$ is denoted by λ .

(b) $t'_i \ll t \ll t_i$: the existence of this regime depends on both the temperature T and the initial order m_0 . The self-consistency of (39) implies $\frac{t'_i}{t_i} < k(k-1)^{(1-k)/k}$. As $T \rightarrow 0$, t'_i tends to zero while t_i does not change. On the other hand, as far as $k > 1$, t_i tends to infinite faster than t'_i as $m_0 \rightarrow 0$. Therefore, this regime exists if m_0 is small but not exactly zero and the temperature is low.

Define a relative order with respect to the equilibrium order,

$$m_r(t, T', m_0) = \frac{m(t) - m(\infty)}{m(\infty)}$$

where $T' = T/T_c$. One can easily obtain from (39)

$$m_r(t) = -\frac{1}{2} \left(\frac{t'_i}{t} \right)^k. \quad (44)$$

The m_r asymptotically follows a power law.

When $t'_i \ll t \ll t_i$ while $t' \ll t'_i$, the response propagator is

$$G_p(t, t') = \Theta(t - t') \left(\frac{t'_i}{t'} \right)^{\theta_T} e^{-\tilde{\lambda} p^\sigma (t-t')}. \quad (45)$$

The time-scale t'_i enters the propagator.

Another case is $t'_i \ll t', t \ll t_i$. The response propagator has an asymptotic form

$$G_p(t, t') = \Theta(t - t') e^{-\tilde{\lambda} p^\sigma (t-t')}. \quad (46)$$

The corresponding equal-time correlation function is

$$C_p(t, t) = \frac{\lambda}{\tilde{\lambda} p^\sigma} (1 - e^{-2\tilde{\lambda} p^\sigma t}). \quad (47)$$

It exponentially tends to the structure function of the equilibrium state.

(c) $t \gg t_i$: notably, in the long-time regime (including all time if $m_0 = 1$), $|m(t)|$ approaches $|m(\infty)|$ from above if $m_0 \neq 0$, and

$$m_r(t) = \frac{1}{2} \left(\frac{t_\mu}{t} \right)^{k-1}. \quad (48)$$

The asymptotic power law has an exponent $\theta'_1 = 1 - k$, which is different from $-\beta/\nu z = \frac{1-k}{2}$ in the critical non-linear relaxation. This is in agreement with the results of [20,30] in the special case of $\sigma = 2$. It can be understood since two processes are governed by different fixed points.

In the general case, $0 < T < T_b$ and $m_0 \neq 0$, the dimension of the order-parameter is scale-dependent. As one has seen in the cases (b) and (c), a more suitable scaling operator is the relative order m_r . There exist two off-critical parameters, T' and m_0 . They will be running *versus* the scale in a scale transformation. Therefore one needs two characteristic functions. The characteristic function for m_0 has been given in (29). The characteristic function for T' can be obtained by requiring t_μ scaling as t , then one can have a general scaling relation,

$$m_r(t, T', m_0) = m_r(b^{-z}t, \epsilon(b, T'), \varphi(b, m_0)). \quad (49)$$

The characteristic function $\epsilon(b, T')$ satisfies the following equation

$$\frac{\epsilon^k(b, T')}{1 - \epsilon(b, T')} = b^{-z(k-1)} \frac{T'^k}{1 - T'}. \quad (50)$$

It can be easily checked that $\epsilon(b, 0) = 0$, $\epsilon(b, 1) = 1$, $\epsilon(1, T') = 1$, and $\epsilon(\infty, T') = 0$. In fact (49) is valid for all temperature (if $m(\infty)$ is replaced by $\sqrt{|\mu|}$ when $T > T_c$) as one will see from the example of the next section. One can again define the effective dimension of T' as

$$y(b, T') = \frac{\ln(\epsilon(b, T')/T')}{\ln(T')}.$$

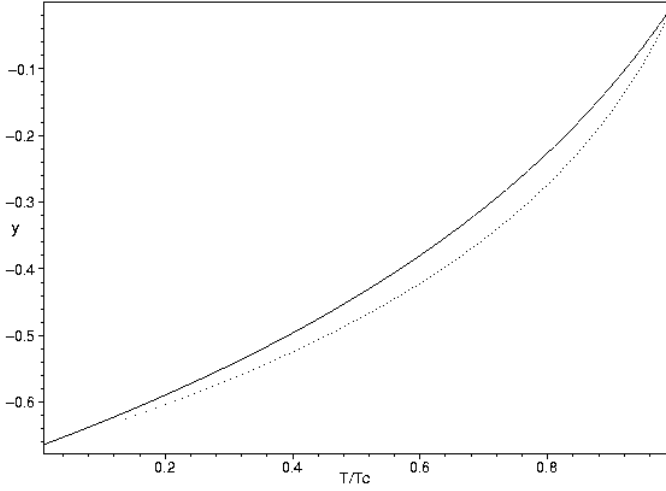


Fig. 2. The effective dimension $y(b, T/T_c)$ versus T/T_c with $k = 3/2$. The upper line is for $b = 2$ while the lower line for $b = 4$.

When $T \rightarrow 0$, one has a constant dimension for T' , *i.e.*, $y(b, 0) = -z \frac{k-1}{k}$. It is negative since $T = 0$ is an attractive fixed point. For $k = 3/2$ and $b = 2, 4$, $y(b, T')$ is plotted versus T' in Figure 2. In the vicinity of the critical temperature, the scaling of the reduced temperature can be recovered from (50), *i.e.*, $\mu(b) = 1 - \epsilon(b, T_c) = b^{z(k-1)}\mu$. The critical exponent $\nu = z(k-1)$ is obtained.

As in the critical dynamics, with the help of the characteristic functions, the response propagator and correlation function can be casted into scaling functions

$$G_p(t, t', T', m_0) = \tilde{h}'(p\xi(t), p\xi(t'), \epsilon(p^{-1}, T'), \varphi(p^{-1}, m_0)) \quad (51)$$

$$C_p(t, t', T', m_0) = p^{-\sigma} \tilde{g}'(p\xi(t), p\xi(t'), \epsilon(p^{-1}, T'), \varphi(p^{-1}, m_0)). \quad (52)$$

Where the characteristic length $\xi(t) \sim t^\rho$ with $\rho = \frac{1}{\sigma}$ is the domain size [1]. Apart from some arbitrary coefficients, \tilde{h}' and \tilde{g}' are universal.

6 Example of $k = 3/2$

In order to see the crossover more clearly, let us consider the concrete example of $k = 3/2$. The 3-dimensional model with the short-ranged interaction belongs to this case. A compact solution for $m(t)$ is available.

(a) $T \leq T_c$: a direct Laplace back-transformation to (23) gives

$$m(t) = m(\infty) \times \left[1 + \frac{2t_i}{\pi t_\mu} \sqrt{\frac{t_\mu}{t}} - \left(1 + \frac{2t_i}{\pi t_\mu} \right) e^{\frac{t}{\pi t_\mu}} \operatorname{erfc}\left(\sqrt{\frac{t}{\pi t_\mu}}\right) \right]^{-1/2} \quad (53)$$

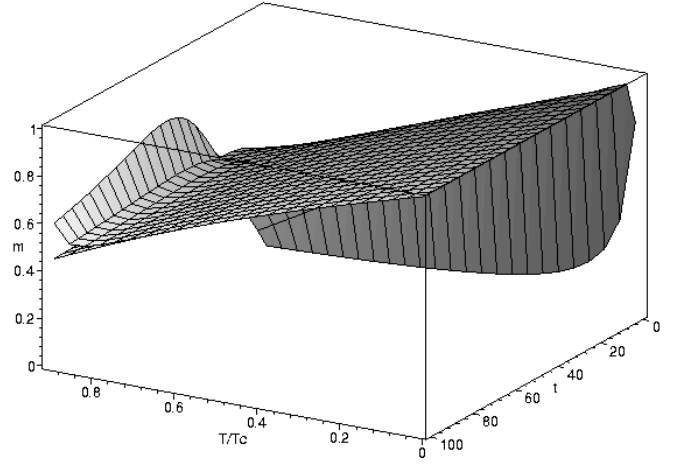


Fig. 3. The time-dependent order $m(t)$ with $k = 3/2$ of (39) and (53). In this figure, t_i has been used as the time scale. The surface has infinite singularity in larger temperatures is the results of (39) obtained by the low-temperature expansion, the other is the exact results of (53).

where $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\tau^2} d\tau$ is the complementary error function.

Taking the limit $\mu = 0$ ($T = T_c$), the critical scaling of (24) for $k = 3/2$ is recovered. In the other hand, for small T , $\mu \sim 1$, one has the ordering scaling for $k = 3/2$ given in Section 5.

Since no expansions around $T = 0$ have been made in obtaining $m(t)$, (53) is valid at temperatures equal and lower than T_c .

In Figure 3, we compare the results of (39, 53). As the self-consistency condition (40) implies, $m(t)$ of (39) gradually deviates from the correct results of (53) as the temperature increases, and becomes singular when it is bigger than certain temperature.

(b) $T > T_c$: for completeness, let us also write down the exponential relaxation in the high temperature phase. Since $\beta_c > \beta$, (23) has a positive singular point $(\pi t_\mu)^{-1}$,

$$F(q) = \frac{1}{|\mu|(\pi t_\mu)^{1/2}} \frac{1 + 2t_i q}{q(q^{1/2} - (\pi t_\mu)^{-1/2})}. \quad (54)$$

The Laplace back-transformation gives

$$m(t) = \sqrt{|\mu|} \times \left[\frac{2t_i}{\pi t_\mu} \sqrt{\frac{t_\mu}{t}} - 1 + \left(1 + \frac{2t_i}{\pi t_\mu} \right) e^{\frac{t}{\pi t_\mu}} \left(2 - \operatorname{erfc}\left(\sqrt{\frac{t}{\pi t_\mu}}\right) \right) \right]^{-\frac{1}{2}}. \quad (55)$$

In the long-time regime, one has the familiar exponential decay

$$m(t) \sim e^{-\frac{t}{2\pi t_\mu}}.$$

7 Conclusion and discussions

In summary, we solve the KSM with long-ranged interactions. The system is quenched from a very high temperature into either $T = T_c$ or $T < T_c$, with an arbitrary m_0 . The bulk order as well as the response propagator and correlation function are calculated.

For small m_0 , we recover the power law of the initial increase in the SCD. We obtain $\theta' = \theta = 1 - \frac{k}{2}$. For the short-ranged interaction, they agree with the existing results.

The initial order increasing power law is also found in the POP, but with the exponent $\theta'_r = k/2$ different from that in the critical dynamics. As the symmetry is broken, a new scaling regime emerges. In the time range $t'_i < t < t_i$ the relative order has a power law $m_r \sim -t^{-k}$, while in the long-time regime $m_r \sim t^{-(k-1)}$.

We obtain the characteristic function of m_0 which enables us to generalize the critical short-time scaling relation to the whole time range and to an arbitrary m_0 .

As in the SCD, the whole ordering process with broken symmetry can be described by universal scaling functions with the help of two characteristic functions. The one that arises from the off-critical temperature is given through the transcendent equation (50). The other one concerning the scaling property of m_0 is exactly the same as in the critical dynamics. This implies that the characteristic function φ does not depend on the dynamics in the model under consideration. If the non-linear part of a dynamics is skipped, one will always obtain a monotonously decreasing effective dimension for the initial order. But it should not be a general conclusion. For the 3-state Potts model, instead a monotonously decreasing function the effective dimension $x_0(m_0)$ firstly increases then goes to zero [9]. Analytical calculations of characteristic functions from the Ising model or the Potts model remain as a challenge.

Although there is a similarity in the short-time behaviors between the SCD and POP, the physical pictures should be quite different for the two processes. In the critical dynamics, the characteristic length is the correlation length which is divergent as $t \rightarrow \infty$. Instead, it is the domain-size that tends to diverge in the POP. The domain-width will be controlled by the temperature. When the temperature approaches T_c from below, the domain-width grows as $t_\mu^{1/z}$. As it gradually become comparable with the domain-size, the phase-ordering picture ceases to be valid. From the consistency condition of the solution for $m(t)$, we obtain the boundary temperature T_b for the ordering scaling domain.

In the specific example of $k = 3/2$, $m(t)$ is obtained for all temperatures. The ordering scaling can smoothly cross over to the critical scaling and then to the high-temperature phase. But at the crossover temperatures, there is no simple scaling law.

We find that the bulk time-dependent order has the special feature that it depends on d and the interaction range-parameter σ only through their ratio. The longer (shorter) the interaction-range is, the higher (lower) the effective dimension. It is physically plausible because the

correlation can be strengthened (weakened) by either increasing (decreasing) the interaction-range or dimension.

The LLNM should have the same properties as those we find in the KSM. Therefore our results about the characteristic functions and the intermediate-time scaling in the POP can be used as the starting-point of finite- n expansions.

We have investigated the KSM in the non-classical regime $d/2 < \sigma < d$. In the classical regime $0 < \sigma < d/2$, the initial increase would be un-observable. When $\sigma \geq d$, the situation would be more complicated. At $\sigma = d$, there would be a logarithmic correction. We leave these for further investigation. Our results can also be extended to a finite-size lattice [13,14].

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